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A NOTE ON TOPOLOGICAL PROPERTIES IN MULTI-VALUED DYNAMICAL SYSTEMS

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Abstract. In this article, we investigate the transitivity and chain transitivity on multi-valued dynamical systems. For compact-valued continuous dynamics, we prove that the notion of transitivity is expressed by the notions of the shadowing property and chain transitivity under locally maximal condition.

1. Introduction

Dynamical systems are worthful tools to interpret a lot of phenomena in the several fields of mathematics, natural science, engineering, social science, and so on. Actually it is common that many theories in dynamical systems give some ideas to solve the problems in the fields. Especially, the multi-valued dynamical systems are closely related to the control theory and provide many applications to the field of engineering. We are interested in the generalization of the theory in dynamical systems to the theory in multi-valued dynamical systems through this article.

The transitive properties in dynamical systems are important to analyze the appearances of orbits [5]. The properties also play meritorious role to study the multi-valued dynamical systems. See [6]. In [4], Chu et al investigated meaningful tools to express the multi-valued dynamical systems. For the systems, they proved that the notion of chain recurrence is equivalent to the notion of nonwandering on compact metric spaces. For noncompact phase spaces, Chu et al [2, 3] investigated the

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dynamic properties of discrete multi-valued dynamical systems. In [7], Sakai investigated about the transitive properties for diffeomorphisms on closed C^∞ manifolds.

In this article, we mainly deal with the transitive properties for compact-valued continuous relations on metric spaces.

Let X be a metric space with a metric d and need not be a compact space. We deal with a relation f on $X \times X$ with $Dom(f) = X$ where $Dom(f) := \{x \in X \mid f(x) \neq \emptyset\}$. For the relation $f \subseteq X \times X$, f is a *compact-valued relation* on X if for each $x \in X$, the image $f(x)$ of x is a compact subset of X. We define the *transpose* f^t of f given by $f^t := \{(y,x) \in X \times X| \ (x,y) \in f\}$. We often use the notation $f^{-1} := f^t$ without confusions. For $x \in X$, a relation f on X is *upper semicontinuous* at x if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x,x_1) < \delta$ implies $f(x_1) \subseteq B(f(x),\varepsilon)$. Here, $B(f(x),\varepsilon)$ is an open ε-ball of the compact set $f(x)$. It is also said that the relation f is *lower semicontinuous* at x if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x,x_1) < \delta$ implies $f(x) \subseteq B(f(x_1),\varepsilon)$. It is naturally defined that the relation f is *continuous* at x if f is both upper semicontinuous at x and lower semicontinuous at x. For details, see [1].

From now on, let f be a compact-valued continuous relation on the metric space X with $Dom(f) = X$.

2. Transitivity and chain transitivity in multi-valued dynamics

Let $\mathcal{P}(X)$ be a collection of all positively real-valued functions defined by

$$\mathcal{P}(X) := \{\varepsilon : X \longrightarrow (0,\infty) \mid \varepsilon \text{ is continuous } \}.$$

For a continuous function $\varepsilon \in \mathcal{P}(X)$, we define two positively real-valued functions $m(\varepsilon,f)$ and $M(\varepsilon,f)$ from X to $(0,\infty)$ given by

$$m(\varepsilon,f)(x) := \min \varepsilon(f(x)) \quad \text{and} \quad M(\varepsilon,f)(x) := \max \varepsilon(f(x))$$

for all $x \in X$. In [3], it is showed that $m(\varepsilon,f), M(\varepsilon,f) \in \mathcal{P}(X)$. For nonempty compact subsets A and B of X, we define *Hausdorff semidistance* $h(A,B)$ of A from B and *Hausdorff distance* $d_H(A,B)$ between A and B given by

$$h(A,B) := \sup\{d(x,B)|x \in A\}$$

and

$$d_H(A,B) := \max\{h(A,B), h(B,A)\}, \text{ respectively.}$$

For a compact-valued continuous relation f and $n \in \mathbb{N}$, the n-th iterated relation f^n of f is also a compact-valued continuous relation on X. Thus we get that for the relation f, if the transpose relation f^{-1} is a compact-valued continuous relation, then the n-th iteration f^n is also compact-valued continuous on X for every $n \in \mathbb{Z}$.

A subset Λ of X is f-invariant if $f(\Lambda) = \Lambda$. A f-invariant subset Λ of X is called *locally maximal* if there exists a compact neighborhood U of Λ such that $\Lambda = \Lambda_f(U)$, where $\Lambda_f(U) = \cap_{n\in\mathbb{Z}} f^n(U)$. We also call that the restriction $f|_\Lambda$ is *locally maximal.* Let $\varepsilon \in \mathcal{P}(X)$. A finite ordered subset $\Psi = (x_0, \cdots, x_n)$ of X is an *ε-chain* for f provided that its length n is at least 1 and that $d_H(x_i, f(x_{i-1})) < m(\varepsilon, f)(x_{i-1})$ for every i with $1 \leq i \leq n$. For a f-invariant set Λ, the restriction $f|_\Lambda$ of f in $\Lambda \times \Lambda$ is defined by $f|_\Lambda(x) := f(x)$ for every $x \in \Lambda$. Now we define transitive properties for the restriction. The restriction $f|_\Lambda$ is *transitive* provided that for every nonempty open subsets V and W, there exists a positive integer n such that $f^n(V) \cap W \neq \emptyset$. The restriction $f|_\Lambda$ is *chain transitive* provided that for every $x, y \in \Lambda$, for every $\varepsilon \in \mathcal{P}(X)$, there exists an ε-chain in Λ from x to y. For a positively real-valued function $\delta \in \mathcal{P}(X)$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ in X is called a *δ-pseudo orbit* if

$$d_H(f(x_i), x_{i+1}) < m(\delta, f)(x_i)$$

for all i in $\mathbb{Z}$. For an $\varepsilon \in \mathcal{P}(X)$, $\{x_i : i \in \mathbb{Z}\}$ is called *ε-traced* by some point $x \in X$ if $d_H(f^i(x), x_i) < m(\varepsilon, f^i)(x)$ for all $i \in \mathbb{Z}$. A compact-valued continuous relation f has *shadowing property* if for every $\varepsilon \in \mathcal{P}(X)$, there exists a $\delta \in \mathcal{P}(X)$ such that any δ-pseudo orbit for f can be ε-traced by some point in X.

LEMMA 2.1. *Let f be a compact-valued continuous relation on X. Let Λ be a closed f-invariant subset in X. Assume that $f|_\Lambda$ is locally maximal and satisfies the shadowing property. Then for any pseudo-orbit of f in Λ, the shadowing point of the psuedo-orbit is in Λ.*

Proof. Let U be a compact neighborhood of Λ satisfying the property that Λ is locally maximal in U. This means that $\Lambda = \Lambda_f(U) = \cap_{n\in\mathbb{Z}} f^n(U)$. From the compactness of Λ, we can take a positive number $d > 0$ such that $B_d(\Lambda) \subseteq U$. Now we consider the constant function $c_d \in \mathcal{P}(X)$ defined by

$$c_d(x) = d \quad \text{for all} \quad x \in X.$$

Since $f|_\Lambda$ has the shadowing property, there exists a positively continuous function $\delta \in \mathcal{P}(X)$ such that every δ-pseudo orbit in Λ is c_d

-shadowed by some point in X. To prove this lemma, we first let a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ of $f|_\Lambda$ in Λ. Since $f|_\Lambda$ has the shadowing property, there exists a shadowing point z in X of the pseudo orbit. So for every integer i,

$$d_H(f^i(z), x_i) < m(c_d, f^i)(z) = \min(c_d(f^i(z))) = d.$$

Thus we get that for every integer i,

$$f^i(z) \subseteq B(x_i, d) \subseteq B(\Lambda, d) \subseteq U.$$

Hence $z \in \cap_{n\in\mathbb{Z}} f^n(U) = \Lambda$ which completes this proof. □

In the next theorem, we prove that the chain transitivity on the multi-valued dynamical systems can be imply to the transitivity on the systems under the shadowing property and locally maximal condition.

THEOREM 2.2. *Let f be a compact-valued continuous relation on X. Let Λ be a closed f-invariant subset in X. Assume that $f|_\Lambda$ is locally maximal and satisfies the shadowing property. If $f|_\Lambda$ is chain transitive, then $f|_\Lambda$ is transitive.*

Proof. Suppose that $f|_\Lambda$ is chain transitive. To prove the transitivity of the relation $f|_\Lambda$, we first let V and W be nonempty open subsets of Λ. So we choose points $x \in V$ and $y \in W$. Then there exists a positive real number d such that $B(x, d) \cap \Lambda \subseteq V$ and $B(y, d) \cap \Lambda \subseteq W$.

Now we consider the constant function $c_d \in \mathcal{P}(X)$ on X which goes to the constant d. Since $f|_\Lambda$ has the shadowing property, there exists a positively continuous function $\delta \in \mathcal{P}(X)$ such that δ-pseudo orbit is c_d -traced by some point. Since $f|_\Lambda$ is chain transitive, we take a δ-chain $\{x_i\}_{i=a}^b$ in Λ such that $x_a = x$ and $x_b = y$. By invarance of $f|_\Lambda$, we pick a positive real orbit of y and a negative real orbit of x under $f|_\Lambda$.

Combining the δ-chain and two real orbits, we can construct a new δ-pseudo orbit in Λ. For the δ-pseudo orbit in Λ, there exists a c_d-tracing point z. By the lemma 2.1, the shadowing point z is in the invariant set Λ. Since for every $i \in \mathbb{Z}$,

$$d_H(f^i(z), x_i) < m(c_d, f^i)(x) = d,$$

we get that

$$f^a(z) \subseteq B(x, d) \subseteq V \text{ and } f^b(z) \subseteq B(y, d) \subseteq W.$$

Hence $f^{b-a}(V) \cap W \neq \emptyset$. Therefore $f|_\Lambda$ is transitive which completes the proof. □

Actually it is obvious that the transitive property satisfies the chain transitive property on the multi-valued dynamical systems. Using the above theorem, we directly obtain the following for the transitvities on the multi-valued dynamical systems.

COROLLARY 2.3. *Let f be a compact-valued continuous relation on X. Let Λ be a closed f-invariant subset in X. If $f|_\Lambda$ is locally maximal and satisfies the shadowing property, then the transivity for $f|_\Lambda$ is equivalent to the chain transitivity for the restriction.*

References

[1] E. Akin, *The General Topology of Dynamical Systems, Graduate Studies in Mathematics*, Amer. Math. Soc. Providence, **71** 1993.
[2] H. Y. Chu, *Chain recurrence for multi-valued dynamical systems on noncompact spaces*, Nonlinear Anal., **61** (2005), 715-723.
[3] H. Y. Chu and J. S. Park, *Attractors for relations in σ-compact spaces.*, Topology Appl., **148** (2005), 201-212.
[4] H. Y. Chu, S. H. Ku and J. S. Park, *Shadowing property on multi-valued dynamical systems*, J. Chungcheong Math. Soc., **29** (2016), 171-176.
[5] C. Conley, *Isolated Invariant sets and the Morse Index C.M.B.S.*, Amer. Math. Soc. Providence, **38** 1978.
[6] R. McGehee, *Attractors for closed relations on compact Hausdorff space*, Indiana Univ. Math. J., **41** (1992), 1165-1209.
[7] K. Sakai, *Shadowable chain transitive sets*, J. Difference Equ. Appl., **19** (2013), 1601-1618.

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